Advanced Quantitative Research Methodology, Lecture Notes: Introduction[1]

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- Some of the best experiences here: getting to know people in other fields

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- Do you have the background for this class? A Test: What's this?

$$b = (X'X)^{-1}X'y$$

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  - We cover different amounts of material each week

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4. Focus, like I will, on learning, not grades: Especially when we work on papers, I will treat you like a colleague, not a student

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- Most important methods originate *outside* the discipline of statistics (random assignment, experimental design, survey research, machine learning, MCMC methods, . . .). Statistics: abstracts, proves formal properties, generalizes, and distributes results back out.

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- Part of a massive change in the evidence base of the social sciences: (a) surveys, (b) end of period government stats, and (c) one-off studies of people, places, or events $\rightsquigarrow$ numerous new types and huge quantities of (big) data

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- This helps us separate the conventions from underlying statistical theory. (How to get an F in Econometrics: follow advice from Psychometrics. Works in reverse too, even when the foundations are identical.)

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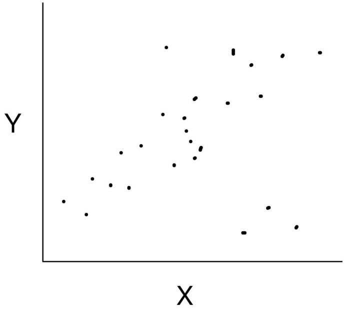

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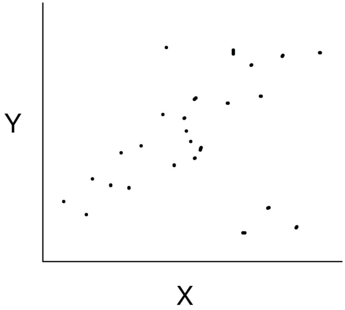

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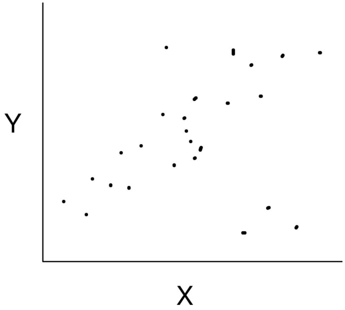

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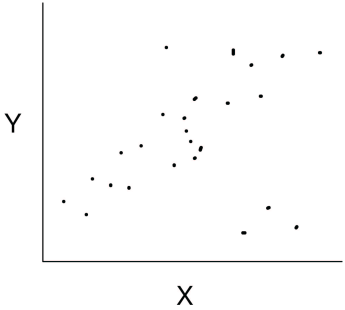

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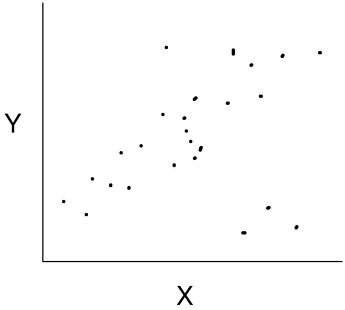

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- from a theory of inference, and for a substantive purpose (like causal estimation, prediction, etc.)

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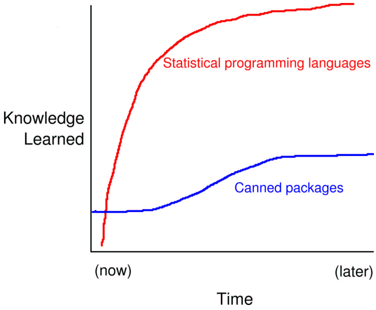

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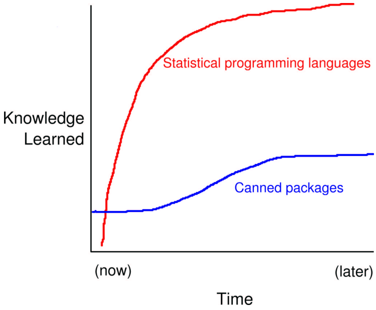

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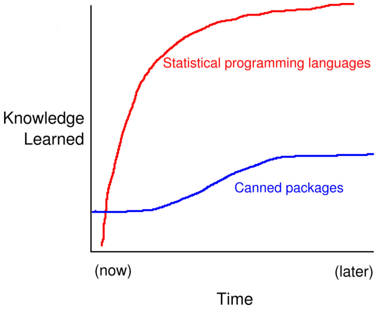

- We'll use R — a free open source program, a commons, a movement
- and an R program called Zelig (Imai, King, and Lau, 2006-14) which simplifies R and helps you up the steep slope fast (see j.mp/Zelig4)

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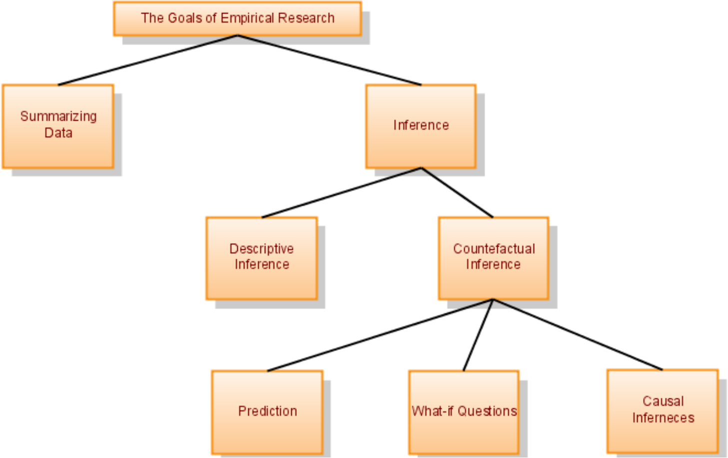

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- Models of dirt on airplanes, vs models of aerodynamics

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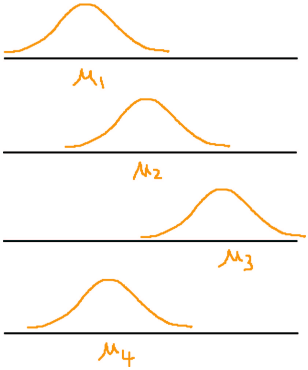

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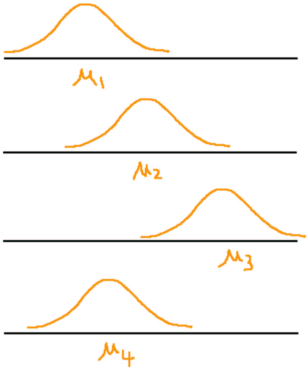

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- (If you know the model, is $R^2 = 1$? Can you predict y perfectly?)

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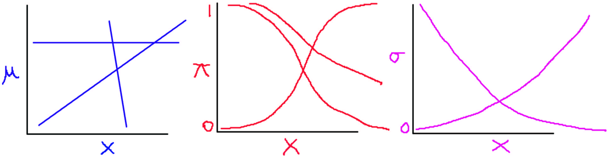

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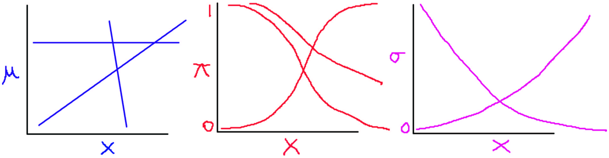

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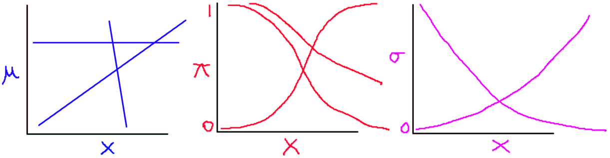

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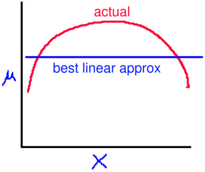

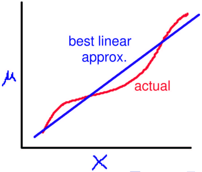

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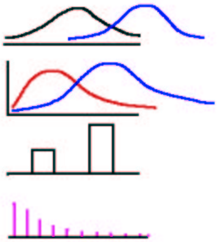

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    - <u>Ameliorate</u> model dependence: preprocess data (via matching, etc.)

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- Rules can be applied *analytically* or via *simulation*.

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5. $\rightsquigarrow$ Empirical evidence: students get the right answer far more frequently by using simulation than math

What is simulation?

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3. The estimate is arbitrarily precise for large n	3. The approximation is arbitrarily precise for large M
4. Example: estimate the mean of the population	4. Example: Approximate the mean of the distribution

Simulation examples for solving probability problems

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alldays <- seq(1, 365, 1)
sameday <- 0
for (i in 1:sims) {
   room <- sample(alldays, people, replace = TRUE)
   if (length(unique(room)) < people) # same birthday
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}
```

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```
Four runs: .538, .550, .547, .524
```

Let's Make a Deal

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In Let's Make a Deal, Monte Hall offers what is behind one of three doors. Behind a random door is a car; behind the other two are goats. You choose one door at random. Monte peeks behind the other two doors and opens the one (or one of the two) with the goat. He asks whether you'd like to switch your door with the other door that hasn't been opened yet. Should you switch?

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for (i in 1:sims) {
  WinDoor <- sample(doors, 1)
  choice <- sample(doors, 1)
  if (WinDoor == choice)                        # no switch
  WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]           # switch
  if (any(doorsLeft == WinDoor))
  WinSwitch <- WinSwitch + 1
}
```

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}
cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
cat("Prob(Car | switch)=", WinSwitch/sims, "\n")
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Let's Make a Deal

Pr(car|No Switch)	Pr(car|Switch)
.324	.676
.345	.655
.320	.680
.327	.673

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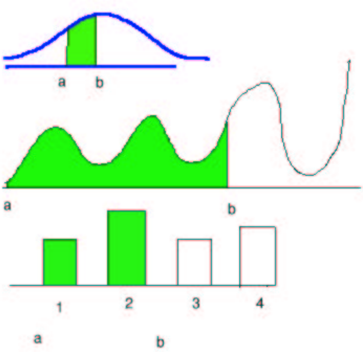
2. $\mathrm{P}(Y) \geq 0$ for every Y

Computing Probabilities from Densities



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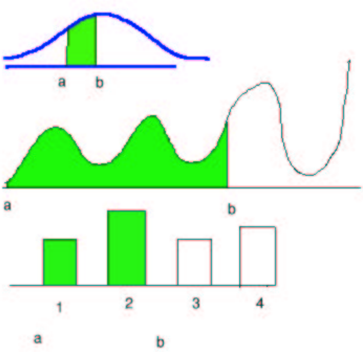

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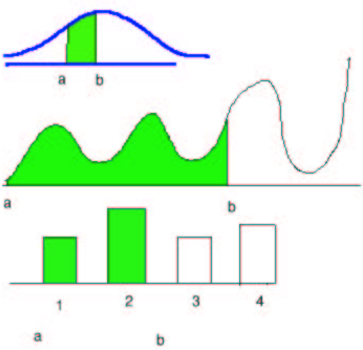

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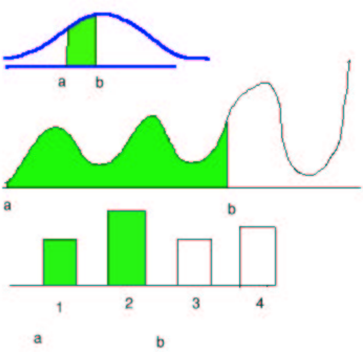

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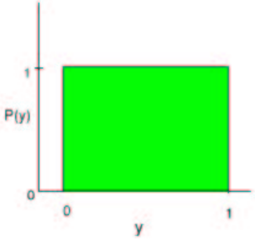

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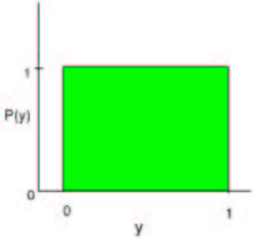

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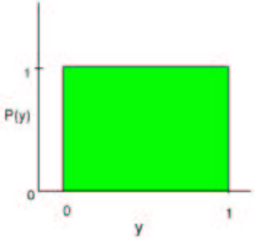

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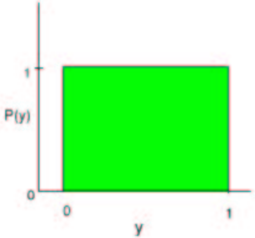

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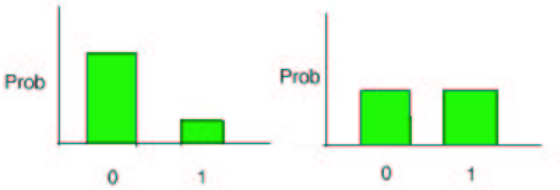
  - $\implies \Pr(Y_i = y|\pi_i) = \pi_i^y (1 - \pi_i)^{1-y}$
  - Alternative notation: $\Pr(Y_i = y|\pi_i) = \text{Bernoulli}(y|\pi_i) = f_b(y|\pi_i)$

Graphical summary of the Bernoulli



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- How do we compute $E(Y^2)$?

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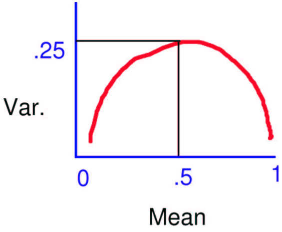

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Discretization for random draws from discrete pmfs

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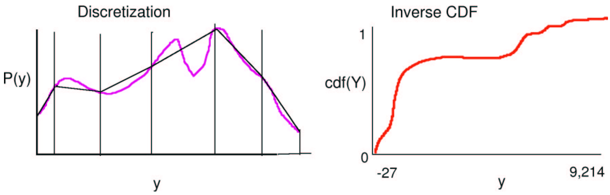

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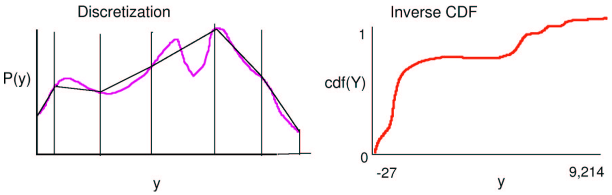

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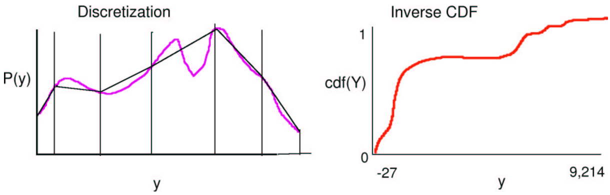

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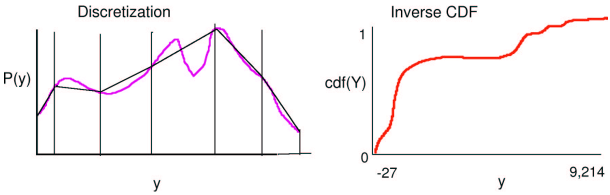

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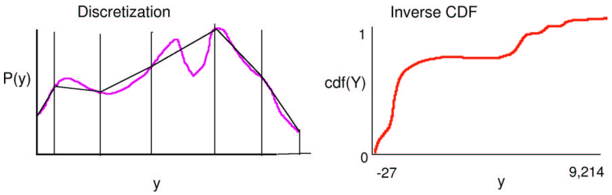

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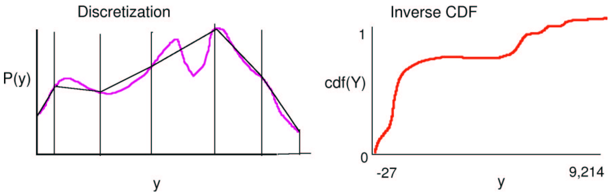

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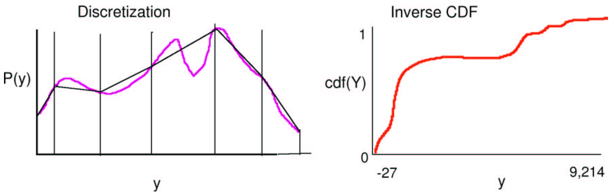

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- (Works for a few dimensions, but Infeasible for many)

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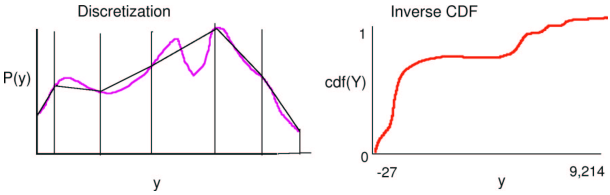

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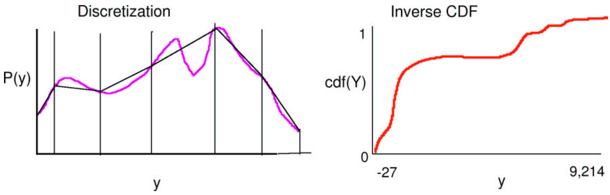

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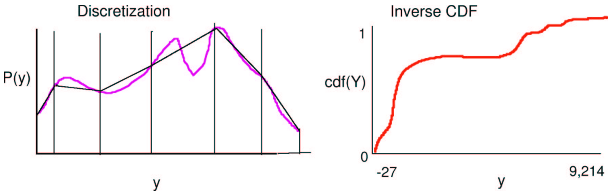

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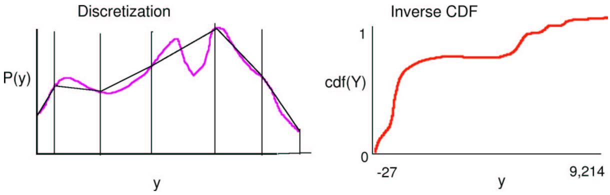

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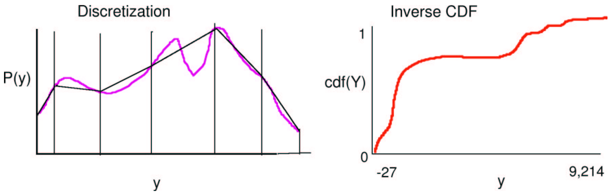

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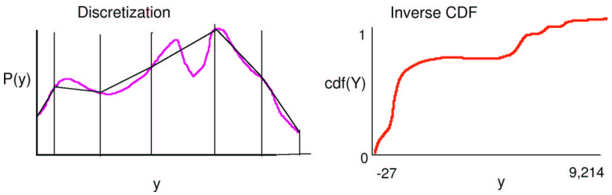

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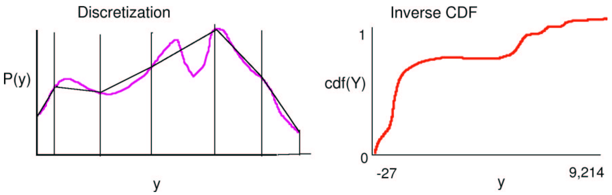

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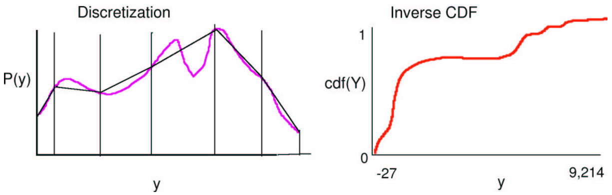

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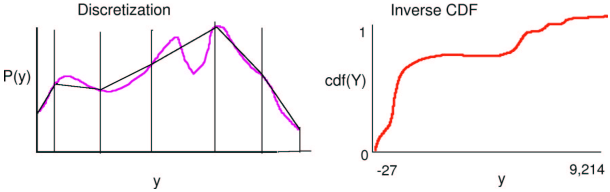

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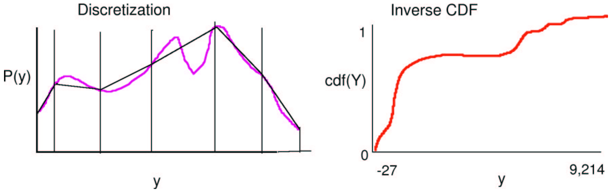

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- Let $Y_i \equiv \{Y_{1i}, \ldots, Y_{ki}\}$ be a $k \times 1$ vector, jointly random:

$$Y_i \sim N(y_i|\mu_i, \Sigma)$$

where μ_i is $k \times 1$ and Σ is $k \times k$. For $k = 2$,

$$\mu_i = \begin{pmatrix} \mu_{1i} \\ \mu_{2i} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

- Mathematical form:

$$N(y_i|\mu_i, \Sigma) = (2\pi)^{-k/2}|\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(y_i - \mu_i)'\Sigma^{-1}(y_i - \mu_i)\right]$$

- Simulating *once* from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the `MASS` library).

Multivariate Normal Distribution

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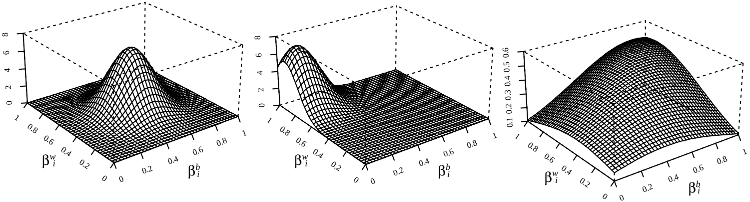
$$N(Y_1|\mu_1, \sigma_1^2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} N(y_i|\mu_i, \Sigma) dy_2 dy_3 \cdots dy_k$$

Truncated bivariate normal examples (for β^b and β^w)



(a) 0.5 0.5 0.15 0.15 0 (b) 0.1 0.9 0.15 0.15 0 (c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1, μ_2, σ_1, σ_2, and ρ.

Stop here

We will stop here this year and skip to the next set of slides.
Please refer to the slides below for further information on probability densities and random number generation; they offer more sophisticated .

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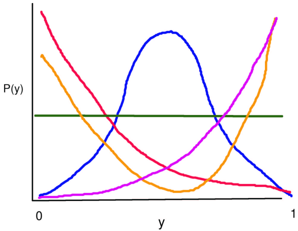

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Reparameterization like this will be key throughout the course.

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- Add up the $\tilde{z}$'s to get $y = \sum_j^N \tilde{z}_j$, which is a draw from the beta-binomial.

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&= \frac{N!}{y_i!(N-y_i)!} \prod_{j=0}^{y_i-1}(\mu+\gamma j) \prod_{j=0}^{N-y_i-1}(1-\mu+\gamma j) \prod_{j=0}^{N-1}(1+\gamma j)
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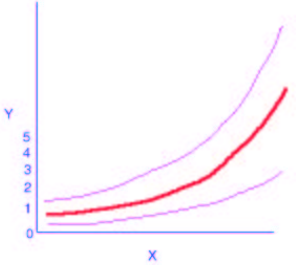

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Negative Binomial Derivation

Recall:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \implies \Pr(AB) = \Pr(A|B)\Pr(B)$$

$$\begin{aligned}
\text{NegBin}(y|\phi, \sigma^2) &= \int_0^\infty \text{Poisson}(y|\lambda) \times \text{gamma}(\lambda|\phi, \sigma^2) d\lambda \\
&= \int_0^\infty \text{P}(y, \lambda|\phi, \sigma^2) d\lambda \\
&= \frac{\Gamma\left(\frac{\phi}{\sigma^2-1} + y_i\right)}{y_i!\Gamma\left(\frac{\phi}{\sigma^2-1}\right)} \left(\frac{\sigma^2 - 1}{\sigma^2}\right)^{y_i} \left(\sigma^2\right)^{\frac{-\phi}{\sigma^2-1}}
\end{aligned}$$